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Non-generic connections corresponding to front solutions

James Powell† and M Tabor‡

Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

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Abstract. A classification of special 'nonlinear' front solutions for certain one-time and one-space reaction–diffusion equations is presented, using the method of Weiss, Tabor and Carnevale (WTC). These results are related to known stability criteria, in particular the steepness criterion of van Saarloos. The WTC method is shown to be equivalent to a special first-order reduction, and both of these methods are shown to work for reaction–diffusion equations with special nonlinearities. Of particular interest is the fact that the special first-order reduction is shown to give separatrices in appropriate phase spaces. An example of a reaction–diffusion equation is presented without these special nonlinearities. While this equation is shown to have a special 'nonlinear' connection and resulting stability properties, it is intractable for either a singular manifold expansion or a first-order reduction. A Lie symmetry analysis is carried out, and it is shown that equations with continuous groups other than tranlational invariance are only a subclass of equations which are amenable to the special solution techniques. However, the 'rescaling ansatz' of Cariello and Tabor suggests that some symmetries are present.

1. Introduction

For years attention has been focused on front solutions to reaction–diffusion equations of the form

$$u_t = u_{xx} + P(u) \tag{1}$$

with

$$P(u) = \sum_{j=1}^{n} p_j u^j .$$

In particular, such equations arise in the theory of first- and second-order phase transitions. A front solution connects any two steady, spatially invariant states with an interface moving at constant speed c. Therefore a front may be written in the form

$$u = u(x - ct).$$

† Permanent address: Department of Mathematics and Statistics, Utah State University, Logan, UT 84322-3900, USA.

‡ Permanent address: Division of Applied Mathematics, Department of Applied Physics, Columbia University, New York, NY 10027, USA.

With the change of variables $z = x - ct$, a front satisfies the ODE

$$u_{zz} + cu_z + P(u) = 0 \tag{2}$$

with boundary conditions

$$\begin{aligned} u &\to u_s \quad \text{as} \quad z \to \pm\infty, \quad P(u_s) = 0 \\ u &\to u_u \quad \text{as} \quad z \to \mp\infty, \quad P(u_u) = 0. \end{aligned}$$

Normally, u_s is taken to be a stable state of (1) at $z = -\infty$, while u_u is taken to be the (unstable) ground state, $u_u = 0$. In this context, a front represents a change between two states, one characterized by $u = u_s$ and the other by the ground state.

Interest in equations in the form of (1) began with the works of Fisher and Kolmogorov [1, 2], who independently examined the equation

$$u_t = u + u_{xx} - u^3 \tag{3}$$

which models the spatial spread of some selective genotype through a population. In (3), $u = 0$ corresponds to a population without the selective adaptation, which is unstable through the process of natural selection. Given even a hint of a new, selective mutation, the corresponding genotype increases in probability and spreads through the population over time. This 'spread' is equivalent to a front solution of (3).

Kolmogorov [2] showed that small, compactly supported initial conditions always evolve into fronts with a certain preferred velocity, $c = c^* = 2$. This means that a selective mutation spreads through a population at a characteristic speed, independent of its initial distribution. Kolmogorov's argument is based on the method of steepest descents, applied to an integral representation of solutions to the linear portion of (3). Dee and Langer [3] and Ben-Jacob *et al* [4] presented an equivalent argument based on considering the unstable modes of $u = 0$ in (3). If

$$u \sim \exp\left[\sigma t + \mathrm{i}k(x - ct)\right]$$

then the linear portion of (3) gives the dispersion relation

$$\sigma = \mathrm{i}ck + 1 - k^2.$$

Dee and Langer conjecture that the mode k^* with maximum growth rate will dominate the development of the far field, and hence determine the asymptotic behaviour of fronts. Maximality gives the condition

$$0 = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right|_{k=k^*} = \mathrm{i}c^* - 2k^* .$$

Secondly, they argue that in the frame of reference travelling with the stable, asymptotic front, the growth rate should be zero, or

$$\sigma(c^*, k^*) = \mathrm{i}c^* k^* + 1 - (k^*)^2 = 0 .$$

These two conditions give

$$(k^*)^2 = -1$$

and

$$c^* = 2$$

for a front connecting $u_s = 1$ at $z = -\infty$ to $u_u = 0$ at $z = \infty$.

One striking feature of the Dee and Langer–Kolmogorov analysis is that it only depends on the *linear* portion of (1). In general, all equations with linear part

$$u_t = \mu u + u_{xx}$$

select the same asymptotic speed $c^* = 2\sqrt{\mu}$ (writing $p_1 = \mu$ in (1)). The form of the nonlinearity plays no role. In direct contrast, van Saarloos [5, 6, 7] presented examples in which an asymptotic speed $\bar{c} \neq c^*$ is chosen by the PDE. In particular, for two equations

$$u_t = u_{xx} + \mu u + u^2 - u^3 \tag{4}$$

and

$$u_t = u_{xx} + \mu u + u^3 - u^5 \tag{5}$$

the special first-order reduction ansatz

$$\frac{\mathrm{d}u}{\mathrm{d}z} = g(u)$$

gives a special, 'nonlinear' solution. This nonlinear solution is steeper, faster, and more stable than the 'linear' Kolmogorov front in some parameter regimes. These results led van Saarloos to present a selection criterion based on steepness. His criterion is that the dynamically preferred front is the front with steepest asymptotic spatial approach to zero. In some parameter regimes, he shows, the linear front is steepest, while in others the nonlinear front is steepest.

Powell *et al* [8] expanded on van Saarloos' work. Firstly they suggested that van Saarloos' steepness criterion is equivalent to a maximum temporal growth rate criterion in the tail of fronts. Secondly, they showed that the nonlinear front is a continuous deformation of the unique front which exists when $\mu < 0$ (for more on these unique subcritical fronts, see Jones *et al* [12]). When the nonlinear front is preferred, it is a non-generic, 'strongly heteroclinic' (SH) connection. By non-generic connection we mean a heteroclinic connection which begins and ends precisely along an eigendirection of its fixed points. In the SH case the connection is to the strongest attracting eigendirection about $u = 0$ in (2). Thirdly, these authors showed that an extension of the methods of Weiss, Tabor, and Carnevale (WTC) give the same unique front. This method is based on a 'constrained' expansion about a 'singular manifold' in complex space-time, and yields a variety of solutions, including fronts, when the expansion is restricted to real space-time (see Carriello and Tabor [9, 10] or Powell *et al* [8] for more details). Cariello and Tabor [9, 10] also showed that this modified WTC method is equivalent to other special solution techniques, such as Hirota's method, for these systems.

In light of the above work the intentions of this paper are fourfold. First, for all equations of the form (1) we will identify the situations in which the WTC approach works and what kinds of phase-space connections it produces. In particular, we note when the method yields strongly heteroclinic connections which result in asymptotic fronts according to the steepness criterion. Secondly, we will show that van Saarloos' first-order reduction [7] and the WTC method are equivalent methods for fronts. This makes the classification of possible connections exhaustive within the class of presently obtainable exact solutions. But more importantly, we will show that the special first-order reduction must always yield separatrix solutions, which are non-generic. It therefore may be possible to capture the skeleton of phase space using either the WTC method or the special first-order reduction.

In section 4, we will introduce an example of a reaction–diffusion PDE for which the WTC method does not work. We will also show that this WTC counter-example has strongly heteroclinic connections in non-trivial parameter regimes, using continuity arguments in the ODE phase space. Numerically we will demonstrate that the special connection dominates in some parameter regimes. Consequently, we show that although the WTC method can select strongly heteroclinic connections in many cases, it cannot capture all of the special connections which exist.

In the final section we will use Lie symmetry methods [11] to try and determine if there are any continuous groups that correspond to the class of WTC-amenable equations. As it turns out, only a subclass of the WTC-amenable equations have non-trivial Lie symmetries. In addition, discrete symmetries such as the $u \to -u$ symmetry do not appear to play any significant role. However, use of a rescaling/resummation extension of the WTC method suggests that certain 'symmetries' may be present.

2. The WTC method for fronts

Cariello and Tabor [9, 10] demonstrated that a modified WTC method can give front solutions to (1) when $P(u) = u - u^3$. Powell *et al* [8] showed that the method also gives front solutions for $P(u) = \mu u + u^3 - u^5$. The basic approach for front connections in (2) is to examine the behaviour of u near a singularity in complex z. We assume that at a pole the solution behaves as

$$u = \frac{a(z)}{[\phi(z)]^{\beta}} \tag{6}$$

where β is a power to be determined and $\phi \to 0$ at the singularity. The use of (6) can be thought of as the extreme truncation of a Laurant expansion about the singular manifold $\phi(z) = 0$. To determine the power β we ask that the largest nonlinearity in (2) balance the leading derivative, which gives

$$\beta(\beta+1)a\,(\phi_z)^2\,\phi^{-\beta-2} \sim p_n a^n \phi^{-n\beta}$$

and hence implies $-\beta - 2 = -n\beta$. This gives the power

$$\beta = \frac{2}{n-1}.$$

Knowing the 'leading balance' allows us to solve (2) following [9, 10]. Separate equations appear at orders $\phi^{-\beta}$, $\phi^{-\beta-1}$ and $\phi^{-\beta-2}$, whence

$$p_j = 0 \qquad \text{for} \qquad j \neq 1, (n+1)/2, n\,. \tag{7}$$

The order of nonlinearity n must either be 2 or odd. Permissable distributions of nonlinearities occur in (3), (4) and (5); another acceptable form is

$$P(u) = \mu u + u^4 - u^7 .$$

Since $P(u)$ can have only three terms, without loss of generality we can always rescale $|p_n| = |p_{(n+1)/2}| = 1$ (provided $p_{(n+1)/2} \neq 0$). We will write $p_1 = \mu$, and note $p_n = -1$ since $\beta > 0$. The canonical form of the WTC-amenable travelling frame ODE is thus

$$u_{zz} + cu_z + \mu u + p_{(n+1)/2} u^{(n+1)/2} - u^n = 0 . \tag{8}$$

Following [9, 10], we find

$$\phi_z = \frac{C_1^{1/\beta}}{\sqrt{\beta(\beta+1)}} e^{(\lambda/\beta)z}$$

with

$$(2\beta + 1)\lambda = \pm p_{(n+1)/2} - c\beta .$$

Solving simultaneous equations for c and λ gives

$$\lambda = \frac{1}{2}\left[\pm\frac{\sqrt{\beta+1}}{2\sqrt{\beta}} p_{(n+1)/2} + (\pm)\sqrt{p_{(n+1)/2}^2 \frac{\beta+1}{4\beta} + \frac{2\mu}{\beta}}\right] \tag{9}$$

$$c = \mp p_{(n+1)/2}\sqrt{\beta+1}\frac{\beta+1}{\sqrt{\beta}} - (\pm)\frac{1}{2}\sqrt{p_{(n+1)/2}^2 \frac{\beta+1}{4\beta} + \frac{2\mu}{\beta}} . \tag{10}$$

With algebraic manipulation and translational invariance, the WTC method gives a solution

$$u = \frac{u_s\,[e^{\alpha z}]^\beta}{[1 + e^{\alpha z}]^\beta} = \frac{\left[\alpha\sqrt{\beta(\beta+1)}e^{\alpha z}\right]^\beta}{[1 + e^{\alpha z}]^\beta} \tag{11}$$

where $\alpha\beta = \lambda$ and u_s satisfies $P(u_s) = 0$.

We may write the WTC solution as

$$u = \left[\frac{\sqrt{\beta(\beta+1)}\phi_z}{\phi}\right]^\beta . \tag{12}$$

Differentiating (12) with respect to z and using the special form of ϕ, we now see that the WTC solution is precisely in the form of van Saarloos' first-order reduction, namely

$$u_z = \beta u\left[\alpha - \frac{1}{\sqrt{\beta(\beta+1)}} u^{(n-1)/2}\right] .$$

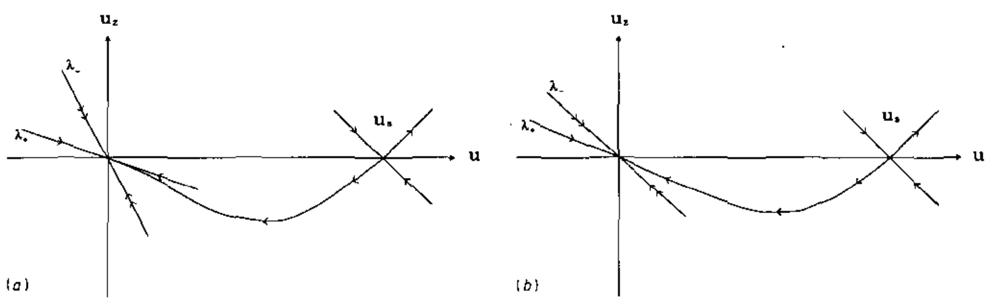
What we are interested in knowing is this: when does (11) correspond to a strongly heteroclinic (SH) trajectory, and therefore an asymptotic front due to the steepness criterion? In general, as $z \to \infty$ for a front solution u

$$u \to K_+(c)e^{\lambda_+ z} + K_-(c)e^{\lambda_- z}$$

where

$$\lambda_\pm(c) = \tfrac{1}{2}\left[-c \pm \sqrt{c^2 - 4\mu}\right] . \tag{13}$$

Consequently the dominant, generic asymptotic behaviour of a front is $u \sim e^{\lambda_+ z}$ as $z \to \infty$. A non-generic (NG) front satisfies either $K_+(c) = 0$ or $K_-(c) = 0$. The SH is an NG connection to the strong eigendirection ($K_+(c) = 0$). When it exists it is steeper than any generic front and is therefore the asymptotic front state. These three distinct situations are illustrated in figure 1.



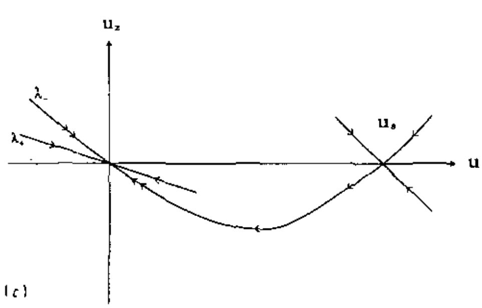


Figure 1. Possible connections between the fixed points $u = u_s$ and $u = 0$ in the phase space of equation (2). (*a*) Generic connection asymptotic to the weaker eigendirection corresponding to λ_+. (*b*) Non-generic (NG) connection directly to the weaker eigendirection. (*c*) Strongly heteroclinic (SH) connection to the strong eigendirection, corresponding to λ_-.

We now set about determining if (11) is SH. Let $\tilde{c}$ denote the speed given by (10). Expanding (11) gives

$$u = u_s\left[e^{\alpha\beta z} - \beta e^{\alpha(\beta+1)} + \cdots\right] .$$

The solution u is SH when

$$\alpha\beta = \lambda_-(\tilde{c})$$

and $\alpha\beta = \lambda_-(\tilde{c})$ (replacing λ with $\alpha\beta$ in (9)) when

$$\frac{p_{(n+1)/2}(2\beta+1)}{-2\sqrt{\beta(\beta+1)}} \pm \frac{1}{2\beta}\left[\frac{\beta p_{(n+1)/2}^2}{(\beta+1)} + \frac{4\beta\mu}{\beta+1}\right]^{1/2}$$
$$= \left[\frac{(2\beta+1)^2+1}{4\beta(\beta+1)} p_{(n+1)/2}^2 + \frac{\mu}{\beta(\beta+1)} \mp \frac{p_{(n+1)/2}}{2\beta\sqrt{\beta(\beta+1)}}\sqrt{\frac{\beta p_{(n+1)/2}^2}{(\beta+1)} + \frac{4\beta\mu}{\beta+1}}\right]^{1/2}. \qquad (14)$$

Here the $\pm$ is the same as $(\pm)$ in equation (9). This gives the possibilities shown in table 1.

Table 1. Summary of possible results.

$p_{(n+1)/2}$	μ	$\pm$ in (9)	Result
0	All	–	NG connection impossible
<0	$\mu < 0$	●	No solution
<0	$\mu > 0$	–	NG connection impossible
>0	$\mu < \mu_c$	–	SH connection, $K_+(\tilde{c}) = 0$
>0	$\mu > \mu_c$	–	NG connection, $K_-(\tilde{c}) = 0$

NG connections are possible only when $p_{(n+1)/2} > 0$, and select the strong eigendirection (are SH) only when

$$\mu < \mu_c = \beta(\beta+1) = \frac{2(n+1)}{(n-1)^2}.$$

This latter condition comes from requiring the LHS of (14) to be positive.

It is easy to test that when $\alpha\beta \neq \lambda_-(\tilde{c})$, $\alpha\beta = \lambda_+(\tilde{c})$. This raises another question, however. When $\alpha\beta = \lambda_+(\tilde{c})$, what about the second term in the series expansion of (11)? Is $\alpha(\beta+1) = \lambda_-(\tilde{c})$, and so does the WTC method choose some precise balance between the linear eigendirections?

We find $\alpha(\beta+1) = \lambda_-(\tilde{c})$ when

$$\left[\frac{(2\beta+1)^2+1}{4\beta(\beta+1)} p_{(n+1)/2}^2 + \frac{\mu}{\beta(\beta+1)} \mp \frac{p_{(n+1)/2}}{2\beta\sqrt{\beta(\beta+1)}}\sqrt{\frac{\beta p_{(n+1)/2}^2}{(\beta+1)} + \frac{4\beta\mu}{\beta+1}}\right]^{1/2}$$
$$= p_{(n+1)/2}\frac{2\beta+3}{2\sqrt{\beta(\beta+1)}} \pm \frac{1}{2\beta}\sqrt{\frac{\beta p_{(n+1)/2}^2}{(\beta+1)} + \frac{4\beta\mu}{\beta+1}}.$$

Squaring both sides and eliminating common terms gives

$$-p_{(n+1)/2}^2 = \pm p_{(n+1)/2}\sqrt{p_{(n+1)/2}^2 + 4\mu}.$$

This can only be satisfied when $p_{(n+1)/2} = 0$ or $\mu = 0$. As we will show in section 5, this corresponds to the class of equations which exhibit non-trivial Lie symmetries. In other classes of equations, when $\alpha\beta = \lambda_+(\tilde{c})$, it turns out that $\alpha(\beta + 1) > \lambda_-(\tilde{c})$. The second linear eigendirection is not involved, since $K_-(\tilde{c}) = 0$ in these cases. Thus, it is fair to say that WTC gives a non-generic heteroclinic connection (in the sense that one of $K_\pm(\tilde{c})$ must be zero) when all three terms of $P(u)$ are non-zero. When $K_+(\tilde{c}) = 0$, these solutions are particularly interesting because the resulting SH front is the asymptotically selected front.

3. Equivalence of WTC and first-order reduction

In the previous section we have shown that the WTC method for fronts is restricted to equations with three terms in $P(u)$, of order u, $u^{(n+1)/2}$ and u^n. We have also shown that the WTC class of solutions is a subclass of solutions given by using a special first-order reduction (SFOR). The question we turn to now is: are any further solutions possible using the SFOR? It seems that the answer is no, and that SFOR and the WTC method work only in the same circumstances. Moreover, in this section we will also show that the SFOR will always give separatrix-type solutions when $p_{(n+1)/2} \neq 0$.

To implement an SFOR, we assume that the front trajectories can be written in the form

$$u_z = g(u) \tag{15}$$

and (2) becomes

$$gg' + cg + P(u) = 0 \tag{16}$$

a new first-order system with independent variable u and dependent variable g. Attempting a series solution to (16), let

$$g = \sum_{k=1}^{\infty} g_k u^k .$$

Substituting into (16) gives a recursion relation at order u^{2k+1}

$$[c + 2(k+1)g_1]\, g_{2k+1} = (k+1) \sum_{j=2}^{2k} g_j g_{2k+1-j} - (k+1) g_{k+1}^2 - p_{2k+1}$$

and at order u^{2k}

$$[c + (2k+1)g_1]\, g_{2k} = \frac{2k+1}{2} \sum_{j=2}^{2k-1} g_j g_{2k-j} - p_{2k} .$$

Here we have used the convention $p_m = 0$ when $m > n$. Notice that if P has reflectional symmetry ($P(u) = -P(-u)$), then all even terms in the expansion for g must be zero. Since the above recursion relations involve products only of even and odd terms in the series for g, the entire summation will be zero for sufficiently large k. From this it follows that the series must truncate.

In the general case, where P may have no special symmetry, the mth coefficient of the power series is growing as $m-2$ if it is non-zero, due to the summation on the RHS of the recursion relations. The resulting series can converge only conditionally in u. $P(u)$ is, however, analytic in u, and therefore we expect no less than global convergence from solutions to (16). Consequently, the series for g must truncate.

Let N be the number of terms in the polynomial for g, i.e.

$$g = \sum_{k=1}^{N} g_k u^k .$$

In general, unless the coefficients of P are chosen in some special fashion, the recursion relations give $N + N - 1 \geqslant n$ independent algebraic equations. However, there are only $N + 1$ unknowns, if we count the speed c as an unknown. Hence, SFOR only works when

$$N + 1 \geqslant N + N - 1$$

or

$$2 \geqslant N \qquad \text{and} \qquad n \leqslant 3 .$$

At this point it is clear that the SFOR cannot work for just any polynomial of order n. However, it has been shown that the SFOR works for equations like (5). Therefore it is possible to choose $P(u)$ so that SFOR works when $n \geqslant 3$. At the very least g must have two terms, g_1 and g_N, with $2N - 1 = n$ and therefore n odd. This results in three equations, at orders u, u^N and u^{2N-1}, for the three unknowns g_1, g_N and c. Adding any other term g_j where $1 < j < N$, must add at least two more equations, resulting in five equations and only four unknowns. From this analysis, it becomes clear that the most general case in which SFOR works is

$$N = \frac{n+1}{2} \qquad g = u\left(g_1 + g_N u^{N-1}\right)$$

and

$$P(u) = \mu u + p_N u^N - u^{2N-1} .$$

These are precisely the same circumstances in which the WTC method gives a solution.

Moreover, SFOR must always give the same solution as WTC. Equation (15) is separable, giving

$$\mathrm{d}z = \frac{\mathrm{d}u}{u\left(g_1 + g_N u^{N-1}\right)} = \frac{\mathrm{d}u}{g_1 u} - \frac{g_N u^{N-2} \mathrm{d}u}{g_1\left(g_1 + g_N u^{N-1}\right)} .$$

Integrating both sides gives

$$g_1 z = \ln(u) - \frac{1}{N-1} \ln\left[g_1 + g_N u^{N-1}\right]$$

and inverting to find u in terms of z gives

$$u = \frac{|g_1| e^{g_1 z}}{\left[1 + g_N e^{(N-1)g_1 z}\right]^{1/(N-1)}} .$$

It is easy to see that this is equivalent to (11) with $\beta = 1/(N-1) = 2/(n-1)$, $\alpha = (N-1)g_1$ and $u_s = \left(|g_1|/g_N\right)^{1/\beta}$.

This analysis shows that the classification given in the previous section is exhaustive among the special solutions we know how to get. The WTC method and SFOR give precisely the same solutions in all circumstances where they work. We will now show that the SFOR must always give a separatrix-type solution, and consequently an NG connection when $p_{(n+1)/2} \neq 0$. However, as we will see in the next section, there are definitely SH connections in equations for which WTC and SFOR are inapplicable, which means that the existence of an SFOR is sufficient, but not necessary, for a NG connection to exist. The connection between SFORs and separatrices can be seen by recognizing that SFOR solution(s) are singular in that they depend on only one, as opposed to two, arbitrary parameters. From the standpoint of the original, second-order ODE, the separatrices correspond to envelope solutions and are thus, by definition, singular.

Let

$$\frac{\mathrm{d}H}{\mathrm{d}u} = 2P(u)$$

with

$$H(0) = 0 .$$

We may rewrite the front ODE (8) in integral form

$$v^2 + 2c \int_{z_0}^{z} v^2 \, \mathrm{d}z = H(u_0) - H(u)$$

where

$$v = \frac{\mathrm{d}u}{\mathrm{d}z}$$

$v(z_0) = 0$ and $u(z_0) = u_0$. This may also be written as

$$-2cv^2 = \frac{\mathrm{d}}{\mathrm{d}z}\left[v^2 + H(u)\right]$$

which is a statement about the monotonic decrease of total 'energy' on the potential surface depicted in figure 2. Using a change of variables, the above expressions become

$$v^2 + 2c \int_{u_0}^{u} v \, \mathrm{d}u = H(u_0) - H(u). \tag{17}$$

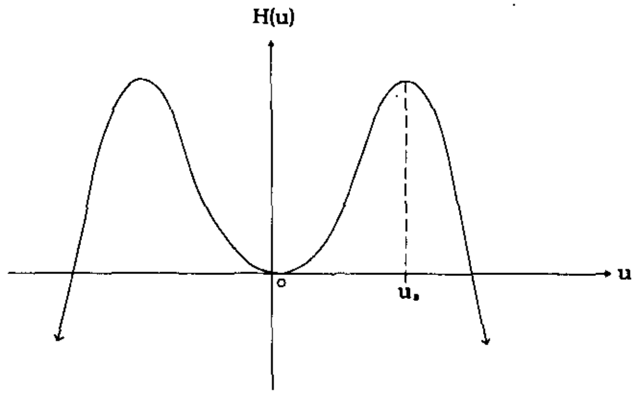


Figure 2. The potential energy, $H(u)$. Front trajectories act like a ball rolling down the potential hill. The front velocity, c, acts as a frictional slowing. One maximum of H is located at the fixed point $u = u_s$.

The integral form of the equation can be written concisely as

$$G(u, v, c, u_0) = 0\,.$$

An envelope solution to (8) is given when

$$\frac{\partial G}{\partial u_0} = 0 \qquad \text{and} \qquad \frac{\partial G}{\partial c} = 0\,.$$

By the inverse function theorem, this happens only for a trajectory on which v and u are functionally related, i.e.

$$v = Q(u)$$

which is, of course, precisely the form of the SFOR (15). This is all true provided

$$\frac{\partial G}{\partial v} \neq 0$$

on the interior of the region for which $v = Q(u)$ is defined.

A number of things now become apparent. Using the the explicit form for G, $\partial_{u_0} G = 0$ when

$$0 = -2cv(u_0) = \frac{\mathrm{d}H(u_0)}{\mathrm{d}u_0} = 2P(u_0)$$

since, by definition, we are setting $v(u_0) = 0$. Thus, any envelope trajectory must include a fixed point of (8) as its initial condition. If we are talking of front solutions, this means that $u_0 = u_s$, since u_s is the positive maximum of the potential H. Hence, the trajectory $v = Q(u)$ must be the unstable manifold of u_s, which must also include the ground state $u = 0$ since the potential 'energy' must always decrease along a trajectory. The condition $\partial_c G = 0$ holds if and only if $v = Q(u)$ and Q has at least as many derivatives as P, locally. Consequently, near $u = 0$ we may write

$$v = q_1 u + \tfrac{1}{2} q_2 u^2 + \cdots + \frac{1}{m!} q_m u^m + \cdots\,. \tag{18}$$

From our discussion of phase space, we also know that near $u = 0$

$$u \sim K_+ e^{\lambda_+ z} + K_- e^{\lambda_- z}$$

whence

$$\begin{aligned} v &= \lambda_+ K_+ e^{\lambda_+ z} + \lambda_- K_- e^{\lambda_- z} + \cdots \\ &= \lambda_+ u + K_- \left(K_+\right)^{-\lambda_-/\lambda_+} (\lambda_- - \lambda_+) u^{\lambda_-/\lambda_+} + \cdots . \end{aligned} \tag{19}$$

This means that one of three things must be true. Either

$$\frac{\lambda_-}{\lambda_+} = m \qquad \text{and} \qquad q_i = 0 \quad \text{for} \quad 1 < i < m$$

so that (18) and (19) coincide locally, or

$$K_+ = 0$$

or

$$K_- = 0 .$$

These latter two requirements mean that the expansion for v will have no term like

$$u^{\lambda_-/\lambda_+} .$$

From the definition of $\lambda_\pm$ we get

$$\frac{\lambda_-}{\lambda_+} = \frac{c^2 - 2\mu + c\sqrt{c^2 - 4\mu}}{2\mu}$$

and hence the first condition is equivalent to

$$c^2 - 2\mu + c\sqrt{c^2 - 4\mu} = 2m\mu .$$

This may be simplified

$$\frac{(m+1)^2}{m} = \frac{c^2}{\mu} ;$$

From our earlier analysis of the SFOR, we know $m = \frac{1}{2}(n+1)$, and therefore the first condition for the existence of the envelope is

$$\frac{c^2}{\mu} = \frac{(n+3)^2}{2(n+1)}$$

which, as we will show in the fifth section, is precisely the condition in which the original ODE yields a non-trivial Lie symmetry, provided $p_{(n+1)/2} = 0$. As was shown in the second section, λ_- is a multiple of λ_+ only when $p_{(n+1)/2} = 0$. Thus, we have shown that when $p_{(n+1)/2} = 0$ the SFOR solution is also the envelope solution, but not a NG connection. On the other hand, when $p_{(n+1)/2} \neq 0$, we have shown that the envelope solution is NG, and since the SFOR is the envelope solution, the SFOR must yield NG connections. Since for any NG connection one of $K_\pm \to 0$, an NG connection separates one kind of phase space motion from another and must therefore be a separatrix.

What remains to be checked is that

$$G_v = \frac{\partial G}{\partial v} \neq 0\,.$$

Since only the LHS of (17) depends on v, we can write

$$G_v = 2v + 2c(u - u_0)\,.$$

$G_v = 0$ only where $v = c(u_0 - u)$, but for a front trajectory $u \leqslant u_0$ and $v < 0$, while $c > 0$ for $\mu \geqslant 0$ in (8). Hence, $G_v = 0$ only at $u = u_s$ and $v = 0$, and this point is included in the front trajectory only in the limit $z \to \infty$. Thus the SFOR

$$v = Q(u)$$

is valid in (8) if and only if it is the envelope solution of front trajectories defined above, and this in turn necessitates that it be the NG connection we are looking for. This shows that the SFOR for fronts and, by virtue of their equivalence, the WTC solution, must result in the NG connections.

4. A strongly heteroclinic counter-example

We have shown that existing methods for finding SH connections work only when the polynomial nonlinearity has a certain form. In this section we will demonstrate that there are SH connections in equations for which WTC method does not appear to be applicable. Thus, nonlinearities which give rise to SH connections do not necessarily allow the WTC method to be applied; these special connections and resulting front dynamics occur more broadly than our current methods can account for.

For our counter-example, let

$$P(u) = \mu u + u^4 - u^5\,. \tag{20}$$

Note that this is not in the class of nonlinearities discussed in the previous section, since $n = 5$ but $4 \neq \frac{1}{2}(n+1) = 3$. Let $\mu \geqslant 0$, $0 < u_u < u_s$ and $P(u_u) = P(u_s) = 0$. The ODE defining front trajectories is

$$u_{zz} + cu_z + \mu u + u^4 - u^5 = 0\,. \tag{21}$$

The linear theory about u_s, u_u and zero is given by (21) linearized about these fixed points

$$\hat{u}_{zz} + c\hat{u}_z + \frac{\mathrm{d}P}{\mathrm{d}u}\hat{u} = 0\,.$$

Here $\hat{u}$ is the perturbation and $\frac{dP}{du} = P'$ is the derivative of P evaluated at one of u_s, u_u or zero. The resulting characteristic equation is

$$\lambda^2 + c\lambda + P' = 0$$

or

$$\lambda = \tfrac{1}{2}\left[-c \pm \sqrt{c^2 - 4P'}\,\right].$$

When $\mu \leqslant 0$, both $u = 0$ and $u = u_s$ are saddle points, since $P'(0)$ and $P'(u_s) < 0$. The point $u = u_u$, on the other hand, is an attracting node, since $P'(u_u) > 0$. This is also clear from a potential energy formulation of (21). Multiplying (21) by $2u_z$ gives

$$\begin{aligned} -2cu_z^2 &= \frac{d}{dz}\left[u_z^2 + \mu u^2 + \tfrac{2}{5}u^5 - \tfrac{1}{3}u^6\right] \\ &= \frac{d}{dz}\left[u_z^2 + H(u)\right] \end{aligned} \qquad (22)$$

which is a statement that total 'energy' along a trajectory must always decrease (for $c > 0$). The graph of $H(u)$ has maxima at $u = 0, u_s$, and a mimimum at $u = u_u$, as shown in figure 3, from which it is clear that $u = 0, u_s$ have both stable and unstable directions, whereas $u = u_u$ is attracting.

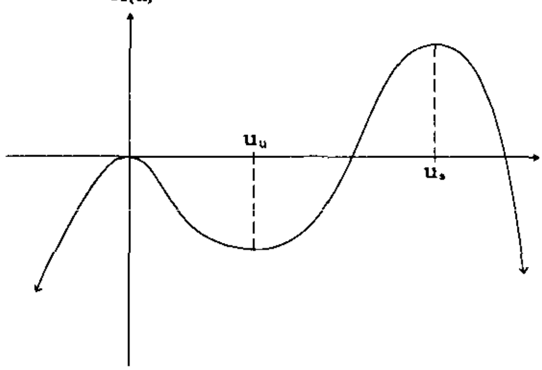


Figure 3. Potential energy, $H(u)$, for the counter-example front problem (25). Note that the potential is not symmetrical, but with $\mu < 0$ there are maxima at $u = 0$ and $u = u_s$.

We can now demonstrate the existence of a trajectory starting at $u = u_s$, ending at $u = 0$, for some particular speed $\bar{c}$. We follow an argument made by Jones, Kapitula, and Powell for a similar equation [12]. Let $w^s(u, c)$ be a trajectory at some speed c such that u decreases to zero as $z \to \infty$, and let $w^u(u, c)$ be a trajectory such that u increases to u_s as $z \to -\infty$. These are illustrated in figure 4. From the decreasing energy statement (22) and figure 3, it is clear that

$$u_u \in w^s \cap w^u \qquad \text{for all} \quad c \geqslant 0$$

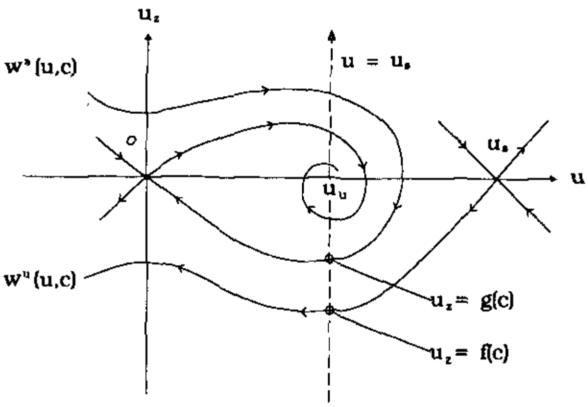


Figure 4. Trajectories in (25) with $c > 0$, so that the energy decreases. The attracting manifold of zero is denoted by w^s and the repelling manifold of $u = u_s$ is denoted by w^u. The functions g and f correspond to the intersections of these manifolds with the line $u = u_u$.

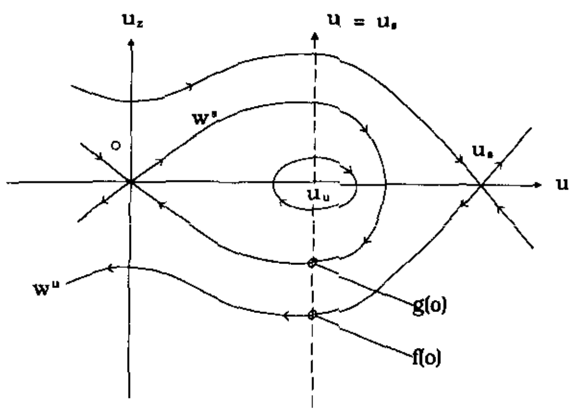


Figure 5. Trajectories in (25) with $c = 0$, so that each path traces out equal-energy surfaces. Note here that $g(0) > f(0)$.

and therefore we may define

$$f(c) = \max\left[u_z; (u_u, u_z) \in w^u(u, c)\right]$$

and

$$g(c) = \min\left[u_z; (u_u, u_z) \in w^s(u, c)\right] .$$

Both f and g are also depicted in figure 4. Since c is a damping constant, it is clear that $f'(c) > 0$ and $g'(c) \leqslant 0$.

Consider the phase space of (21), pictured in figure 5 when $c = 0$. Each trajectory in figure 5 is a contour of equal energy, and comparing the contours which contain $u = 0$ and $u = u_s$ shows that

$$g(0) > f(0)$$

when μ is close enough to zero that $V(u_s) > V(0)$. On the other hand, when c is chosen large enough, say $c = c^\dagger \gg 0$, a trajectory which begins at $u = u_s$ must decrease monotonically to u_u. The damping is too great for the solution to climb the potential well at all, and such trajectories are depicted in figure 6. However $f'(c) > 0$ implies that

$$f(c^\dagger) > f(0) > g(c^\dagger) = 0.$$

Consequently there is a $c = \tilde{c}$ such that $f(\tilde{c}) = g(\tilde{c})$, and uniqueness is given by the sign of the derivatives of f and g. Since different trajectories cannot cross in phase space, we have

$$w^s(u, \tilde{c}) = w^u(u, \tilde{c})$$

and therefore an SH connection must exist when $\mu \leqslant 0$.

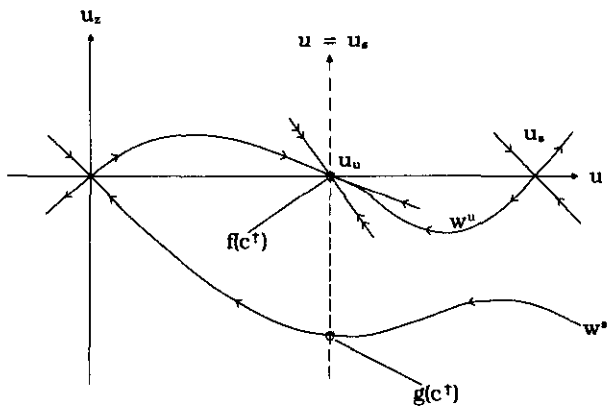


Figure 6. Trajectories in (25) for large $c = c^\dagger$. Subjected to enough damping, a trajectory leaving $u = u_s$ is captured monotonically by the minimum of H, $u = u_u$. This shows that $f(c^\dagger) = 0$, while clearly $g(c^\dagger) \ll 0$. Somewhere w^s and w^u must have crossed.

More importantly, this SH connection must be structurally stable. If we append the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}c = 0$$

to (21) so that the effect of varying c can be included explicitly, we have an autonomous system of equations for the dependent variables u and c. In the resulting phase space, w^s and w^u are centre stable and unstable manifolds, respectively, and the above argument with the derivatives of f and g show that these two manifolds intersect transversely. Because transverse intersection is stable structurally, perturbing the parameter μ cannot destroy the connection. Thus, as we increase μ from negative through zero and to positive, we get a continuous deformation of the original SH trajectory for $\mu < 0$, a sequence of events depicted in figure 7. Consequently, there

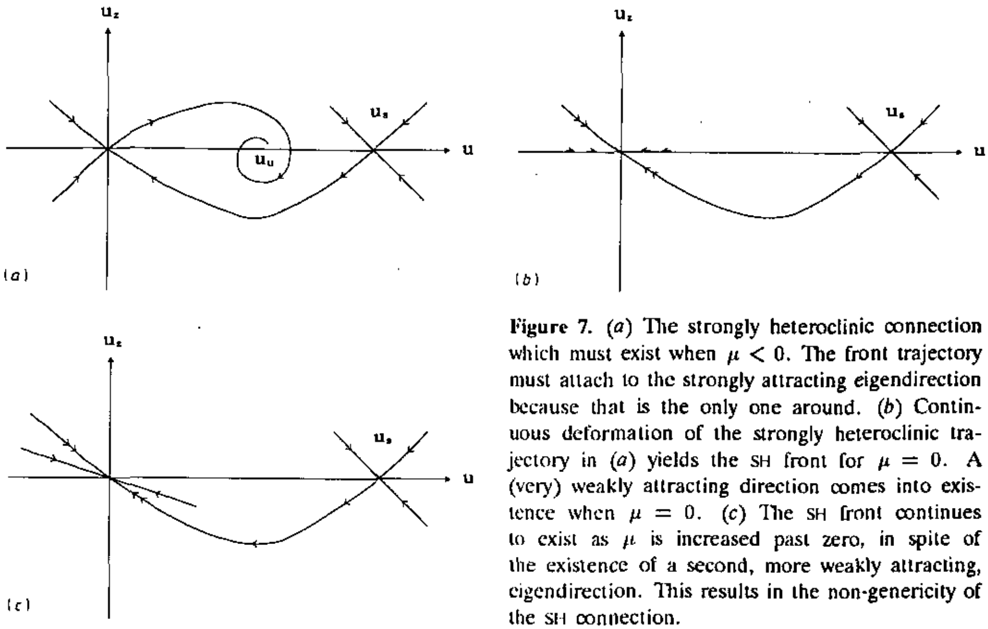


Figure 7. (*a*) The strongly heteroclinic connection which must exist when $\mu < 0$. The front trajectory must attach to the strongly attracting eigendirection because that is the only one around. (*b*) Continuous deformation of the strongly heteroclinic trajectory in (*a*) yields the SH front for $\mu = 0$. A (very) weakly attracting direction comes into existence when $\mu = 0$. (*c*) The SH front continues to exist as μ is increased past zero, in spite of the existence of a second, more weakly attracting, eigendirection. This results in the non-genericity of the SH connection.

exists a $\mu_c > 0$ so that when $\mu < \mu_c$ SH connections exist in (21). This results in five types of possible phase spaces for $0 < \mu < \mu_c$, two of which are degenerate, and the relationships of these phase spaces as c is varied are illustrated in figure 8.

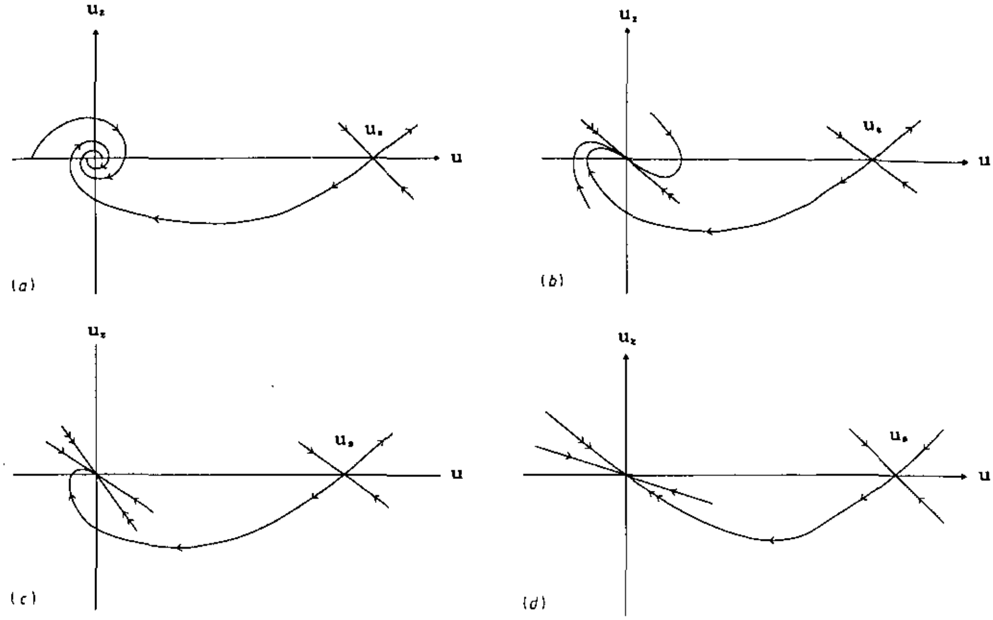
The existence of SH connections in (21) is supported by numerical simulations of the PDE (1) with nonlinearity (20). As described in the introduction, linear theory predicts an asymptotic speed for fronts of $c^* = 2\sqrt{\mu}$, and no asymptotic speed at all when $\mu < 0$. Using the Fourier-in-space, Runge–Kutta in time numerical method described in Powell, Newell and Jones [8] we solved (1) with the initial condition $u(x,0) = 2\mathrm{e}^{-x^2}$, and diagnosed the local speed of the resulting fronts by evaluating

$$c_{\mathrm{local}}(t) = \left.\frac{u_t}{u_x}\right|_{u=u_0}$$

numerically. Here u_0 was chosen $u_0 = 0.1$, and we looked at the final spatial position where $u = u_0$, corresponding to the speed of a right-moving front.

In the first simulation, we choose $\mu = -0.1$ to demonstrate that a front solution does indeed exist when $\mu < 0$. Figure 9 is a plot of u against space, plotted at times $t = 3n$, $n = 0, \ldots, 5$. It is clear from the graph of succeeding slices of the front that the velocity is positive for negative μ sufficiently close to zero.

In the second simulation we chose μ small and positive, $\mu = 0.09$. Here the linear theory predicts an asymptotic speed of $c^* = 2\sqrt{0.09} = 0.6$. Figure 10 is the same as figure 9, except for the difference $\mu = 0.09$. Figure 11 is a plot of $c_{\mathrm{local}}(t)$ for the duration of the simulation. Note that c_{local} converges nicely, but to some $c = \tilde{c} > c^* = 0.6$. This illustrates both that the SH front exists in this parameter regime and that it is an asymptotic state. Figure 12 is a plot of $c_{\mathrm{local}}(t)$ for $\mu = 1$, which is above threshold for the onset of linear fronts. Notice that here $c_{\mathrm{local}} \to c^* = 2$.



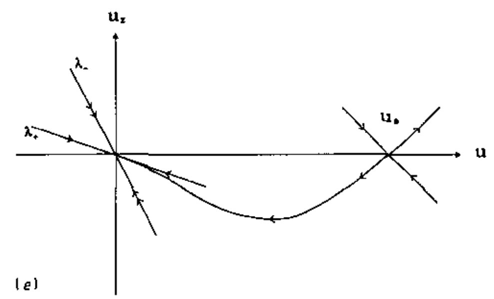


Figure 8. The complete picture of possible front behaviours as the front speed is varied, for $0 < \mu < \mu_c$. This sequence of behaviours was first noticed by Jones [12] and was discussed in detail by Powell [8]. (*a*) For $c < 2\sqrt{\mu} = c^*$ the fronts are oscillatory. (*b*) At $c = c^*$ the phase space is degenerate, and the front approaches zero from the negative side, tangential to the degenerate eigendirection. (*c*) When $c^* < c < \tilde{c}$, the front has generic behaviour and approaches zero along the weaker direction, but it does so from the negative side. (*d*) At $c = \tilde{c}$ the SH front exists. (*e*) For $c > \tilde{c}$ the fronts are once again generic, and approach zero from the positive side.

As far as we know, there is no analytic method for finding the SH front with speed $\tilde{c}$ and non-generic behaviour. In fact, because of the non-generic behaviour it is even hard to find these fronts numerically in (21), since one must look for a tangential intersection of two one-dimensional manifolds, neither one of which is attracting. Of course, a shooting method could determine $\tilde{c}$, if not the form of the front, and then the linear dispersion relation would determine the asymptotic approach to zero by evaluating $\lambda_-(\tilde{c})$. This is enough to determine numerically the cutoff value μ_c above which the linear fronts are preferred using the steepness criterion from the introduction. Nevertheless, any hope that the WTC method, SFOR, or some related methods might prove to be 'magic bullets' for determining SH connections appears to be unfounded. In the previous section we showed that these methods only work for particular sets of nonlinearities in $P(u)$, and in this section we have shown that SH connections exist outside of the class of equations with these kinds of nonlinearities. In the next section we will apply classical methods for determining Lie symmetries to try and understand what is so special about the WTC-amenable equations of the form (8).

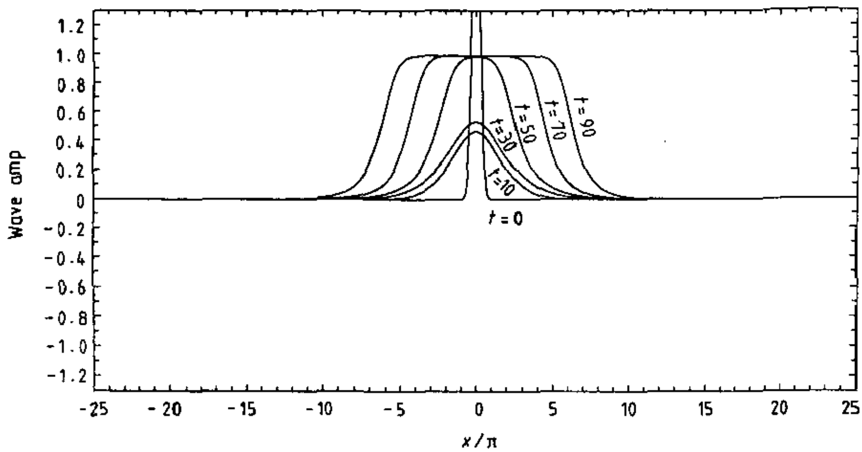


Figure 9. Subcritical fronts in the counter-example (25), for $\mu = -0.1$. The plot is of front amplitude at several different times, versus x measured in multiples of π. The speed is positive, as proven in section 4, and this numerically confirms the existence of an SH connection in (25), since both zero and $u = u_s$ are hyperbolic points.

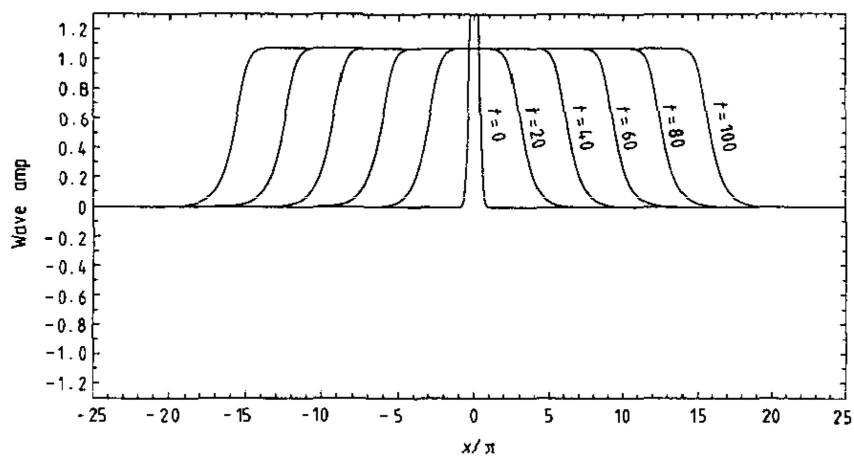


Figure 10. The strongly heteroclinic fronts which exist as the asymptotic front state for $\mu = 0.09 < \mu_c$. Fronts are plotted for several different times in amplitude versus x in multiples of π. The initial condition is Gaussian, while the fronts which result converge to a speed greater than the linear marginal speed $c^* = 0.6$.

5. Symmetries and fronts

We have seen that the reaction–diffusion equation

$$u_t = \mu u + u_{xx} + p_{(n+1)/2} u^{(n+1)/2} - u^n \tag{23}$$

exhibits, in certain parameter regimes, fronts corresponding to strong heteroclinic connections. These connections can be found using the WTC method, or equivalently the van Saarloos SFOR. It is natural to ask if there are any symmetry properties of (23) that might reveal, *a priori*, the possibility of a strong connection. As will be

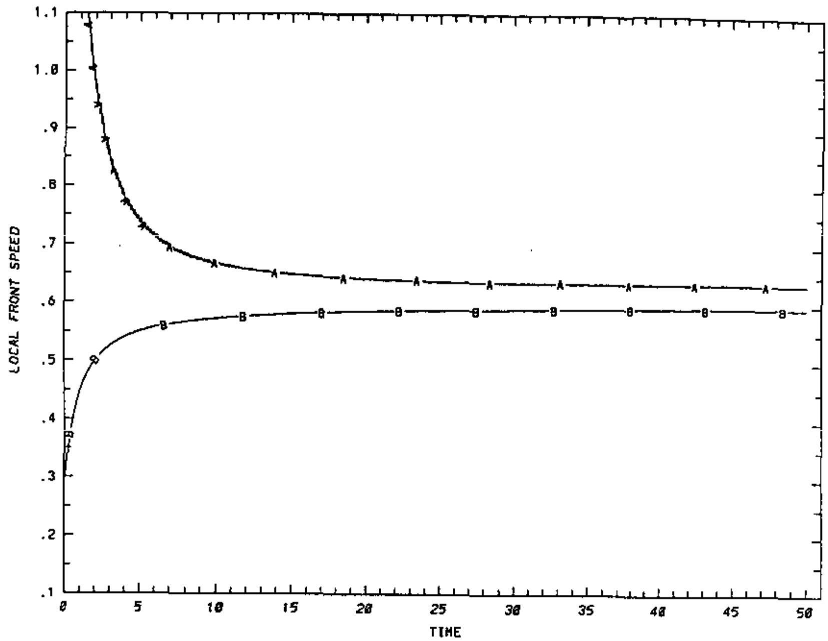


Figure 11. Behaviour of local (in time) front speeds for $\mu = 0.09$ (represented by curve A). Curve B converges to c^* as the inverse of time, while the diagnosed speeds clearly converge to some $\bar{c} > c^* = 0.6$. Again, this demonstrates that SH fronts exist for the counter-example system.

described below, traditional symmetry methods yield little insight into this question, although a recent extension to the WTC method (Cariello and Tabor [10]) does show how the SFOR is related to a type of similarity transformation.

Application of standard Lie symmetry methods to (23) shows that the system only admits a trivial two parameter group corresponding to tranlation in x and t, i.e. the equation is invariant under the transformations $x_1 = x + a\epsilon$, $t_1 = t + b\epsilon$ and $u_1 = u$, where a and b are constants and ϵ is the group parameter. This confirms the fact that $z = x - ct$ is a group invariant, from which follows the standard travelling wave reduction

$$u_{zz} + cu_z + \mu u + p_{(n+1)/2}u^{(n+1)/2} - u^n = 0 .$$

Such a group invariance is an obvious, if not only, prerequisite for the existence of travelling wave fronts.

The only additional symmetry exhibited by (8) arises when $p_{(n+1)/2} = 0$. (However, for the case $n = 3$, an additional symmetry is found for one special value of p_2—we do not pursue this special case here.) When $p_{(n+1)/2} = 0$, we introduce the standard infinitesimal transformations

$$z_1 = z + \epsilon\xi(u, z) + O(\epsilon^2) \qquad \text{and} \qquad u_1 = u + \epsilon\eta(u, z) + O(\epsilon^2)$$

where ϵ is the group parameter and ξ and η are accorded the standard forms

$$\xi(u, z) = f(z) \qquad \text{and} \qquad \eta(u, z) = g(z)u .$$

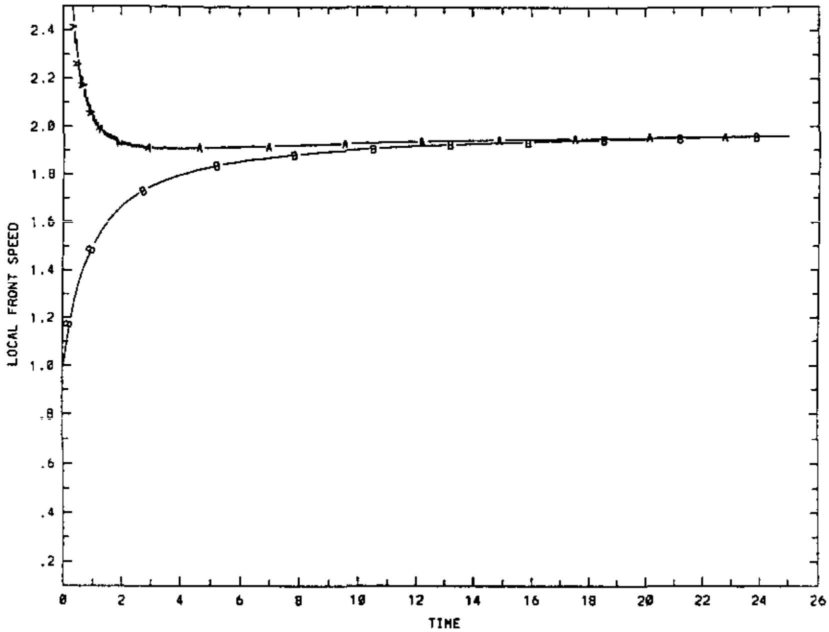


Figure 12. Local behaviour (in time) of front speeds for the Gaussian initial condition when $\mu = 1 > \mu_c$. In this case, the SH front does not exist, and the speeds converge nicely to the marginal speed $c^* = 2$. Curve A represents diagnosed local front speeds, while curve B is a reference curve which converges to c^* as the inverse of time.

It is straightforward to show that f and g must satisfy the equations

$$2g' - f'' + cf' = 0 \qquad g'' + cg' + 2\mu f' = 0 \qquad g = \left(\frac{2}{1-n}\right) f' .$$

From these equations one can deduce that

$$\xi(u, z) = a\mathrm{e}^{\gamma z} + b \tag{24}$$

and

$$\eta(u, z) = -u\delta\gamma\mathrm{e}^{\gamma z} . \tag{25}$$

Here a and b are constants and

$$\gamma = c\left(\frac{n-1}{n+3}\right) \qquad \text{and} \qquad \delta = \frac{2c}{n+3}$$

provided that

$$\frac{c^2}{\mu} = \frac{(n+3)^2}{2(n+1)} .$$

This condition for the existence of the non-trivial group ($a \neq 0$) is the same condition for equation (8) to have the Painlevé property. (The case $a = 0$, $b \neq 0$ corresponds to the usual translational invariance.)

The global transformations associated with (24) and (25) are (setting $a = 1$, $b = 0$)

$$z_1 = z - \frac{1}{\gamma} \log (1 - \epsilon\gamma e^{\gamma z}) \tag{26}$$

and

$$u_1 = u(1 - \epsilon\gamma e^{\gamma z})^{-\delta/\gamma} . \tag{27}$$

Eliminating ϵ between these reveals the invariant combination $ue^{\delta z}$. This suggests introduction of the transformation

$$u = Ae^{-\delta z} F(e^{-\gamma z})$$

which reduces (8) to

$$F'' - \frac{A^{N-1}}{\gamma^2} F^n = 0 \tag{28}$$

where $'$ denotes differentiation with respect to the variable $e^{-\gamma z}$. As nice as this additional symmetry is, it clearly has little to do with the existence of strongly heteroclinic connections, since the case of $p_{(n+1)/2} = 0$ is precisely when such connections do not exist. (This symmetry might, however, provide a rationalization for the observation, made in the third section, that only when $p_{(n+1)/2} = 0$ does the Taylor-expanded WTC solution choose a balance between the two linear eigendirections.) Below we shall show that the transformation leading to (28) is contained naturally in the WTC expansion.

Recent work by Cariello and Tabor [10] has shown that there are connections between the WTC method for non-integrable evolution equations and certain symmetries. This comes about because the constrained singular manifold used in the WTC method can be shown to play the role of a similarity variable. Although this idea was introduced for PDE, it can also be carried through on the ODE (8). Use of the truncated expansion (6) throws away a lot of useful information—especially that contained in certain arbitrary coefficients that can be found (for the constrained ϕ) at higher orders of ϕ. Some of this information can be recaptured by making the ansatz (the 'rescaling ansatz')

$$u = \frac{a(z)}{\phi(z)^\beta} f(\phi) \tag{29}$$

which corresponds to a partial resummation of the full WTC expansion. Substituting (29) into (8) and introducing the variable $F(\phi) = f(\phi)/\phi^\beta$ yields

$$\phi_z [F'' - \beta(\beta + 1) F^n] + p_{(n+1)/2} \sqrt{(\beta + 1)/\beta} \left[F' + \beta F^{(n+1)/2}\right] = 0 . \tag{30}$$

The special form of ϕ means that ϕ_z can be replaced by $\alpha(\phi - 1)$, thereby making (30) a non-autonomous ODE with ϕ as the independent variable.

Equation (30) has some interesting properties. Firstly, note that it has the special pole solution

$$F(\phi) = \phi^{-\beta}$$

which returns the known front solution (11). Secondly, in the limit $t \to \infty$ (recall that z is the travelling wave variable $z = x - ct$), ϕ_z vanishes and (30) reduces to

$$F' + \beta F^{(n+1)/2} = 0\,.$$

Recalling that $u = aF$, this can be transformed back to the original variables to yield

$$u_z - \alpha\beta u + \frac{\beta}{\beta + 1} u^{(n+1)/2} = 0$$

which is exactly the van Saarloos SFOR. Thus this reduction can be seen to correspond to an asymptotic limit of a type of 'symmetry' transformation. Traditional group methods tell us that only when $p_{(n+1)/2} = 0$ is $ue^{\delta z}$ a group invariant; whereas the rescaling ansatz reveals that this is still a useful combination even when $p_{(n+1)/2} \neq 0$ and the particular group is apparently absent. In the case $p_{(n+1)/2} = 0$, our reduction simplifies to

$$F'' - \beta(\beta + 1)F^n = 0$$

which is equal to (28) when an appropriate choice of A is made. Thus, in some sense the 'symmetries' inherent in the rescaling version of the WTC analysis include the traditional, continuous groups.

6. Conclusion

The previous section shows that none of the 'normal' continuous or discrete symmetries seem to give rise to the equations for which the WTC method works. However, a 'similarity' analysis based on the WTC approach seems to capture a kind of 'asymptotic' symmetry of which we have no understanding. We have classified that subclass of equations of the type (1) for which known methods work, and within that class presented a tabulation of when the resulting front solutions are non-generic and, in particular, strongly heteroclinic (SH). The SH structure and its resulting dynamics exist outside of this class. Our results are therefore somewhat negative; we know what current methods cannot do for equations of the form (1), and we know that neither the discrete or continuous symmetries give rise to equations of the form (8). However, our 'negative' results clearly define some extremely interesting questions. What symmetries, if any, *are* encapsulated in equations like (8)? Why should a constrained singular manifold expansion have anything to do with topological properties in phase space? Do similar results hold for 'kink' and 'soliton' solutions obtained using the WTC approach? And can the WTC approach be extended to work outside of the special class we have described? We now know that the WTC trajectories are not merely integrable artifacts in non-integrable systems; they can be dynamically important. Therefore, understanding these trajectories will show us something deep and unknown about the physics of non-integrable systems.

On the 'positive' side, we have shown that the SFOR and hence the WTC method do capture separatrices and SH connections when they exist. This in itself is pleasing, because separatrices are the skeleton of phase space. In the context of reducing infinite-degree systems (like PDE) to finite-order systems (like the ODE we have dealt with here), these solutions are important organizing structures in the dynamical behaviour of the PDE. ODE separatrices act like fixed points in the infinite-dimensional phase space of the PDE, and even if they are unstable they still direct the behaviour of the transients along their unstable manifolds. With a method in hand to find these functional fixed points, it is possible to develope a more complete analytic understanding of non-integrable, nonlinear PDEs.

Acknowledgments

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